

DISCRETE KORN'S INEQUALITY FOR SHELLS

SHENG ZHANG

ABSTRACT. We prove Korn's inequalities for Naghdi and Koiter shell models defined on spaces of discontinuous piecewise functions. They are useful in study of discontinuous finite element methods for shells.

KEY WORDS. Korn's inequality, Naghdi shell, Koiter shell, discontinuous finite elements.

SUBJECT CLASSIFICATION. 65N30, 46E35, 74S05.

1. INTRODUCTION

In the Naghdi shell model, the strain energy is a sum of bending, membrane, and transverse shear strain energies. The three strains are expressed in terms of the model primary variables comprising displacement of the shell mid-surface and rotation of normal fibers, in which derivatives and function values are combined together in a complicated manner by the curvature tensors and Christoffel symbols. Korn's inequality for shells [5] establishes an equivalence between the strain energy norm of the primary variables and the their usual Sobolev norms, which ensures the wellposedness of the the Naghdi shell model in Sobolev spaces. The situation for the Koiter shell model is similar, which excludes the transverse shear effect in shell deformation and only uses mid-surface displacement as the primary variables, for which a Korn's inequality is also available [5]. When analyzing conforming finite element methods for shells [2, 4, 8], such continuous version of Korn's inequalities play a fundamental role in that they attribute most of the numerical analysis and estimates to that in Sobolev spaces. To deal with discontinuous Galerkin methods for shells [16, 17], it is desirable to have appropriate generalizations of the Korn's inequalities that can be applied to piecewise defined discontinuous functions. It seems that such discrete Korn's inequalities do not trivially follow from the Korn's inequalities for shells. Methods of proving discrete Korn's inequality for plane elasticity [6] can not be easily adapted to shell problems either. It is the purpose of this paper to establish such discrete Korn's inequalities for both the Naghdi and Koiter shell models, for the most general geometry of shell mid-surfaces, without making any additional assumption on the regularity of shell model solutions.

Department of Mathematics, Wayne State University, Detroit, MI 48202 (szhang@wayne.edu).

Let $\tilde{\Omega} \subset \mathbb{R}^3$ be the middle surface of a shell of thickness 2ϵ . It is the image of a two-dimensional domain $\Omega \subset \mathbb{R}^2$ through a mapping Φ that has third order continuous derivatives. The Naghdi shell model is a two-dimensional model defined on Ω . In the following, we use super scripts to indicate contravariant components of tensors, and use subscripts to indicate covariant components. Greek super and subscripts take their values in $\{1, 2\}$, and Latin scripts take their values in $\{1, 2, 3\}$. Summation rules with respect to repeated and super and subscripts will also be used. We use $H^\alpha(\tau)$ to denote the L^2 based Sobolev space of order α of functions defined on τ , in which the norm and semi-norm are denoted by $\|\cdot\|_{k,\tau}$ and $|\cdot|_{k,\tau}$, respectively. We also use $H^0(\tau)$ to denote $L^2(\tau)$. When $\tau = \Omega$, we simply write the space as H^α .

The coordinates $x_\alpha \in \Omega$ furnish the curvilinear coordinates on $\tilde{\Omega}$ through the mapping Φ . We assume that at any point on the surface, along the coordinate lines, the two tangential vectors $\mathbf{a}_\alpha = \partial\Phi/\partial x_\alpha$ are linearly independent. The unit vector $\mathbf{a}_3 = (\mathbf{a}_1 \times \mathbf{a}_2)/|\mathbf{a}_1 \times \mathbf{a}_2|$ is normal to $\tilde{\Omega}$. The triple \mathbf{a}_i furnishes the covariant basis on $\tilde{\Omega}$. The contravariant basis \mathbf{a}^i is defined by the relations $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta_\beta^\alpha$ and $\mathbf{a}^3 = \mathbf{a}_3$, in which δ_β^α is the Kronecker delta. It is obvious that \mathbf{a}^α are also tangent to the surface. The metric tensor has the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, the determinant of which is denoted by a . The contravariant components are given by $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$. The curvature tensor has covariant components $b_{\alpha\beta} = \mathbf{a}_3 \cdot \partial_\beta \mathbf{a}_\alpha$, whose mixed components are $b_\beta^\alpha = a^{\alpha\gamma} b_{\gamma\beta}$. The symmetric tensor $c_{\alpha\beta} = b_\alpha^\gamma b_{\gamma\beta}$ is called the third fundamental form of the surface. The Christoffel symbols are defined by $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}^\gamma \cdot \partial_\beta \mathbf{a}_\alpha$, which are symmetric with respect to the subscripts.

The Naghdi shell model [15] uses the covariant components u_α of the tangential displacement vector $\mathbf{u} = u_\alpha \mathbf{a}^\alpha$, coefficient w of the normal displacement $w \mathbf{a}^3$ of the shell mid-surface, and the covariant components θ_α of the normal fiber rotation vector $\boldsymbol{\theta} = \theta_\alpha \mathbf{a}^\alpha$ as the primary variables. The bending strain tensor, membrane strain tensor, and transverse shear strain vector associated with a deformation represented by such a set of primary variables are

$$(1.1) \quad \rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) - \frac{1}{2}(b_\alpha^\gamma u_{\gamma|\beta} + b_\beta^\gamma u_{\gamma|\alpha}) + c_{\alpha\beta} w,$$

$$(1.2) \quad \gamma_{\alpha\beta}(\mathbf{u}, w) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w,$$

$$(1.3) \quad \tau_\alpha(\boldsymbol{\theta}, \mathbf{u}, w) = \partial_\alpha w + b_\alpha^\gamma u_\gamma + \theta_\alpha.$$

Here, the covariant derivative of a vector u_α is defined by

$$(1.4) \quad u_{\alpha|\beta} = \partial_\beta u_\alpha - \Gamma_{\alpha\beta}^\gamma u_\gamma.$$

The loading forces on the shell body and upper and lower surfaces enter the shell model as resultant loading forces per unit area on the shell middle surface, of which the tangential force density is $p^\alpha \mathbf{a}_\alpha$ and transverse force density $p^3 \mathbf{a}_3$. Let the boundary $\partial\tilde{\Omega}$ be divided to

$\partial^D \tilde{\Omega} \cup \partial^S \tilde{\Omega} \cup \partial^F \tilde{\Omega}$. On $\partial^D \tilde{\Omega}$ the shell is clamped, on $\partial^S \tilde{\Omega}$ the shell is soft-simply supported, and on $\partial^F \tilde{\Omega}$ the shell is free of displacement constraint and subject to force or moment only. (There are 32 different ways to specify boundary conditions at any point on the shell boundary, of which we consider the three most typical.) Let $\mathbf{H}^1 = H^1 \times H^1$. The shell model is defined in the Hilbert space

$$(1.5) \quad H = \{(\boldsymbol{\phi}, \mathbf{v}, z) \in \mathbf{H}^1 \times \mathbf{H}^1 \times H^1; v_\alpha \text{ and } z \text{ are 0 on } \partial^D \Omega \cup \partial^S \Omega, \\ \text{and } \theta_\alpha \text{ is 0 on } \partial^D \Omega\}.$$

The model determines $(\boldsymbol{\theta}, \mathbf{u}, w) \in H$ such that

$$(1.6) \quad \frac{1}{3} \int_{\Omega} a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}(\boldsymbol{\theta}, \mathbf{u}, w) \rho_{\alpha\beta}(\boldsymbol{\phi}, \mathbf{v}, z) \sqrt{a} dx_1 dx_2 \\ + \epsilon^{-2} \left[\int_{\Omega} a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\mathbf{u}, w) \gamma_{\alpha\beta}(\mathbf{v}, z) + \kappa \mu \int_{\Omega} a^{\alpha\beta} \tau_\alpha(\boldsymbol{\theta}, \mathbf{u}, w) \tau_\beta(\boldsymbol{\phi}, \mathbf{v}, z) \right] \sqrt{a} dx_1 dx_2 \\ = \int_{\Omega} (p^\alpha v_\alpha + p^3 z) \sqrt{a} dx_1 dx_2 + \int_{\partial^S \tilde{\Omega}} r^\alpha \phi_\alpha d\tilde{s} + \int_{\partial^F \tilde{\Omega}} (q^\alpha v_\alpha + q^3 z + r^\alpha \phi_\alpha) d\tilde{s} \\ \forall (\boldsymbol{\phi}, \mathbf{v}, z) \in H.$$

Here, q^i and r^α are the force resultant and moment resultant on the shell edge [15]. The factor κ is a shear correction factor. The last two integrals on the shell edge is taken with respect to the arc length of the boundary of $\tilde{\Omega}$. The fourth order contravariant tensor $a^{\alpha\beta\gamma\delta}$ is the elastic tensor of the shell, defined by

$$a^{\alpha\beta\gamma\delta} = \mu(a^{\alpha\gamma} a^{\beta\delta} + a^{\beta\gamma} a^{\alpha\delta}) + \frac{2\mu\lambda}{2\mu + \lambda} a^{\alpha\beta} a^{\gamma\delta}.$$

Here, λ and μ are the Lamé coefficients of the elastic material. It satisfies the condition that there are constants C_1 and C_2 that only depend on the shell mid-surface and the Lamé coefficients of the shell material such that for any tensor $\varsigma_{\alpha\beta}$

$$\sum_{\alpha, \beta=1}^2 |\varsigma_{\alpha\beta}|^2 \leq C_1 a^{\alpha\beta\lambda\gamma} \varsigma_{\alpha\beta} \varsigma_{\lambda\gamma}, \quad a^{\alpha\beta\lambda\gamma} \varsigma_{\alpha\beta} \varsigma_{\lambda\gamma} \leq C_2 \sum_{\alpha, \beta=1}^2 |\varsigma_{\alpha\beta}|^2.$$

Also, there are constants C_1 and C_2 that only depend on the shell mid-surface such that for any vector ς_α ,

$$\sum_{\alpha=1}^2 |\varsigma_\alpha|^2 \leq C_1 a^{\alpha\beta} \varsigma_\alpha \varsigma_\beta, \quad a^{\alpha\beta} \varsigma_\alpha \varsigma_\beta \leq C_2 \sum_{\alpha=1}^2 |\varsigma_\alpha|^2.$$

These inequalities together with a Korn's inequality [5] assures that the Naghdi shell model (1.6) has a unique solution in the space H . The Korn's inequality states that there is a C

that could be dependent on the shell mid-surface such that

$$\begin{aligned}
 (1.7) \quad & \|\boldsymbol{\theta}\|_{\mathbf{H}^1} + \|\mathbf{u}\|_{\mathbf{H}^1} + \|w\|_{H^1} \\
 & \leq C \left[\sum_{\alpha,\beta=1}^2 \|\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w)\|_{L^2}^2 + \sum_{\alpha,\beta=1}^2 \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{L^2}^2 + \sum_{\alpha=1}^2 \|\tau_{\alpha}(\boldsymbol{\theta}, \mathbf{u}, w)\|_{L^2}^2 + f^2(\boldsymbol{\theta}, \mathbf{u}, w) \right]^{1/2} \\
 & \quad \forall (\boldsymbol{\theta}, \mathbf{u}, w) \in \mathbf{H}^1 \times \mathbf{H}^1 \times H^1.
 \end{aligned}$$

Here f is a continuous seminorm satisfying the rigid body motion condition that if $(\boldsymbol{\theta}, \mathbf{u}, w)$ defines a rigid body motion and $f(\boldsymbol{\theta}, \mathbf{u}, w) = 0$ then $(\boldsymbol{\theta}, \mathbf{u}, w) = 0$. The displacement functions $(\boldsymbol{\theta}, \mathbf{u}, w) \in \mathbf{H}^1 \times \mathbf{H}^1 \times H^1$ defines a rigid body if and only if $\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w) = 0$, $\gamma_{\alpha\beta}(\mathbf{u}, w) = 0$, and $\tau_{\alpha}(\boldsymbol{\theta}, \mathbf{u}, w) = 0$ [5].

We assume that Ω is a bounded polygon. Let \mathcal{T}_h be a shape regular, but not necessarily quasi-uniform, triangulation on Ω . Shape regularity of triangulations is a crucial notion in this paper. It is worthwhile to recall its definition here. Considering a triangle, we let r and R be the radii of its inscribed circle and circumcircle, respectively. Then the ratio R/r is called its shape regularity constant, or simply shape regularity. For a triangulation, the maximum of shape regularities of all its triangles is called the shape regularity of the triangulation [9], denoted by \mathcal{K} . We will need to consider a (infinite) class of triangulations. For a class, the *shape regularity* \mathcal{K} is the supreme of all the shape regularities of its triangulations. For the triangulation \mathcal{T}_h , we use \mathcal{T}_h to denote the set of all (open) triangular elements of the partition, and use Ω_h to denote the union of all the open triangular elements. We use \mathcal{E}_h^0 to denote the set of all interior (open) edges and \mathcal{E}_h^∂ all boundary edges, and let $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$. We use h_τ to denote the diameter of an element $\tau \in \mathcal{T}_h$ and use h_e to denote the length of an edge $e \in \mathcal{E}_h$.

Let H_h^1 be the space of piecewise H^1 functions subordinated to the triangulation \mathcal{T}_h . A function in H_h^1 is independently defined on every element $\tau \in \mathcal{T}_h$ on which it belongs to $H^1(\tau)$. A function u in H_h^1 is certainly in $L^2(\Omega_h)$. On an edge $e \in \mathcal{E}_h^0$, a function u may have two different traces from the two elements sharing e . We use $\llbracket u \rrbracket$ to denoted the absolute value of difference of the two traces, which is the jump of u over e . In the space H_h^1 , we define a norm

$$(1.8) \quad \|u\|_{H_h^1} := \left[\sum_{\tau \in \mathcal{T}_h} \|u\|_{1,\tau}^2 + \sum_{e \in \mathcal{E}_h^0} \frac{1}{h_e} \int_e \llbracket u \rrbracket^2 ds \right]^{1/2}.$$

Let $\mathbf{H}_h^1 = H_h^1 \times H_h^1$. We prove the discrete Korn's inequality for Naghdi shell that for all $(\boldsymbol{\theta}, \mathbf{u}, w) \in \mathbf{H}_h^1 \times \mathbf{H}_h^1 \times H_h^1$

$$(1.9) \quad \|\boldsymbol{\theta}\|_{\mathbf{H}_h^1}^2 + \|\mathbf{u}\|_{\mathbf{H}_h^1}^2 + \|w\|_{H_h^1}^2 \leq C \left[\sum_{\alpha, \beta=1}^2 \|\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w)\|_{0, \Omega_h}^2 + \sum_{\alpha, \beta=1}^2 \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{0, \Omega_h}^2 + \sum_{\alpha=1}^2 \|\tau_\alpha(\boldsymbol{\theta}, \mathbf{u}, w)\|_{0, \Omega_h}^2 + f^2(\boldsymbol{\theta}, \mathbf{u}, w) + \sum_{e \in \mathcal{E}_h^0} \frac{1}{h_e} \int_e \sum_{\alpha=1}^2 ([\theta_\alpha]^2 + [u_\alpha]^2) \, ds + \sum_{e \in \mathcal{E}_h^0} \frac{1}{h_e} \int_e [w]^2 \, ds \right].$$

Here f is a seminorm such that $f(\boldsymbol{\theta}, \mathbf{u}, w) \leq C(\|\boldsymbol{\theta}\|_{\mathbf{H}_h^1} + \|\mathbf{u}\|_{\mathbf{H}_h^1} + \|w\|_{H_h^1})$ and it satisfies the rigid body motion condition that if $(\boldsymbol{\theta}, \mathbf{u}, w)$ defines a rigid body motion and $f(\boldsymbol{\theta}, \mathbf{u}, w) = 0$ then $(\boldsymbol{\theta}, \mathbf{u}, w) = 0$. The constant C could be dependent on the shell mid-surface and the shape regularity \mathcal{K} of the triangulation \mathcal{T}_h , but otherwise is independent of the triangulation. We shall simply say that such a constant is independent of \mathcal{T}_h .

The Koiter shell model [13] uses the covariant components u_α of the tangential displacement $u_\alpha \mathbf{a}^\alpha$ and the coefficient w of the normal displacement $w \mathbf{a}^3$ of the shell mid-surface as the primary variable. Such a displacement deforms the surface $\tilde{\Omega}$ and changes its curvature and metric tensors. The linearized change in curvature tensor is the bending strain tensor. It is expressed in terms of the displacement components as

$$(1.10) \quad \rho_{\alpha\beta}^K(\mathbf{u}, w) = \partial_{\alpha\beta}^2 w - \Gamma_{\alpha\beta}^\gamma \partial_\gamma w + b_{\alpha|\beta}^\gamma u_\gamma + b_\alpha^\gamma u_{\gamma|\beta} + b_\beta^\gamma u_{\gamma|\alpha} - c_{\alpha\beta} w.$$

The linearized change of metric tensor is the membrane strain tensor $\gamma_{\alpha\beta}(\mathbf{u}, w)$ that has the same expression as in Naghdi model (1.2). There is no transverse shear in the Koiter model. Indeed the bending strain tensor (1.10) can be obtained from the bending strain tensor of the Naghdi model (1.1) by eliminating the rotation vector $\boldsymbol{\theta}$ using the zero-shear condition that $\tau_\alpha(\boldsymbol{\theta}, \mathbf{u}, w) = 0$, and multiplying the result by -1 . I.e.,

$$\rho_{\alpha\beta}^K(\mathbf{u}, w) = -\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w) \text{ with } \theta_\alpha = -\partial_\alpha w - b_\alpha^\beta u_\beta.$$

As in the Naghdi model, we let the boundary $\partial\tilde{\Omega}$ be divided to $\partial^D\tilde{\Omega} \cup \partial^S\tilde{\Omega} \cup \partial^F\tilde{\Omega}$. On $\partial^D\tilde{\Omega}$ the shell is clamped, on $\partial^S\tilde{\Omega}$ the shell is simply supported, and on $\partial^F\tilde{\Omega}$ the shell is free of displacement constraint and subject to force or moment only. (There are 16 different ways to specify boundary conditions at any point on the shell boundary, of which we consider the three most typical.) The shell model is defined in the Hilbert space

$$(1.11) \quad H^K = \{(\mathbf{v}, z) \in \mathbf{H}^1 \times H^2 \mid v_\alpha \text{ and } z \text{ are 0 on } \partial^D\Omega \cup \partial^S\Omega, \text{ and the normal derivative of } z \text{ is 0 on } \partial^D\Omega\}.$$

The model determines $(\mathbf{u}, w) \in H^K$ such that

$$\begin{aligned}
 (1.12) \quad & \frac{1}{3} \int_{\Omega} a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}^K(\mathbf{u}, w) \rho_{\alpha\beta}^K(\mathbf{v}, z) \sqrt{a} dx_1 dx_2 + \epsilon^{-2} \int_{\Omega} a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\mathbf{u}, w) \gamma_{\alpha\beta}(\mathbf{v}, z) \sqrt{a} dx_1 dx_2 \\
 & = \int_{\Omega} (p^\alpha v_\alpha + p^3 z) \sqrt{a} dx_1 dx_2 + \int_{\partial^S \tilde{\Omega}} m D_{\tilde{\mathbf{n}}} z d\tilde{s} + \int_{\partial^F \tilde{\Omega}} (q^\alpha v_\alpha + q^3 z + m D_{\tilde{\mathbf{n}}} z) d\tilde{s} \\
 & \qquad \qquad \qquad \forall (\mathbf{v}, z) \in H^K.
 \end{aligned}$$

Here, q^i and m are resultant loading functions on the shell boundary, which can be calculated from force resultants and moment resultants on the shell edge [13]. The scalar z can be viewed as defined on $\tilde{\Omega}$. We let $\tilde{\mathbf{n}} = \tilde{n}^\alpha \mathbf{a}_\alpha$ be the unit outward normal to $\partial\tilde{\Omega}$ that is tangent to $\tilde{\Omega}$. The derivative $D_{\tilde{\mathbf{n}}} z := \tilde{n}^\alpha \partial_\alpha z$ is the directional derivative in the direction of $\tilde{\mathbf{n}}$ with respect to arc length. The elastic tensor $a^{\alpha\beta\gamma\delta}$ is the same as in Naghdi model. The wellposedness of the Koiter model (1.12) hinges on a Korn's inequality for Koiter shell [5] that there is a constant C such that

$$\begin{aligned}
 (1.13) \quad & \|\mathbf{u}\|_{\mathbf{H}^1} + \|w\|_{H^2} \leq C \left[\sum_{\alpha,\beta=1,2} \|\rho_{\alpha\beta}^K(\mathbf{u}, w)\|_{L^2}^2 + \sum_{\alpha,\beta=1,2} \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{L^2}^2 + f^2(\mathbf{u}, w) \right]^{1/2} \\
 & \qquad \qquad \qquad \forall \mathbf{u} \in \mathbf{H}^1, w \in H^2.
 \end{aligned}$$

Here $f(\mathbf{u}, w)$ is a semi-norm on $\mathbf{H}^1 \times H^2$ that satisfies the rigid body motion condition that if (\mathbf{u}, w) defines a rigid body motion and $f(\mathbf{u}, w) = 0$ then $\mathbf{u} = 0$ and $w = 0$. A displacement $(\mathbf{u}, w) \in \mathbf{H}^1 \times H^2$ defines a rigid body motion of the shell mid-surface if there are constant vectors \mathbf{c} and \mathbf{d} such that $u_\alpha \mathbf{a}^\alpha + w \mathbf{a}^3 = \mathbf{c} + \mathbf{d} \times \Phi(x_1, x_2)$. This is equivalent to $\rho_{\alpha\beta}^K(\mathbf{u}, w) = 0$ and $\gamma_{\alpha\beta}(\mathbf{u}, w) = 0$ [5].

For the Koiter model, on the triangulation \mathcal{T}_h , we also need to consider piecewise H^2 functions that are independently defined on each element, with the norm defined by

$$(1.14) \quad \|w\|_{H_h^2} := \left[\sum_{\tau \in \mathcal{T}_h} \|w\|_{2,\tau}^2 + \sum_{e \in \mathcal{E}_h^0} \left(\sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket \partial_\alpha w \rrbracket^2 ds + h_e^{-1} \int_e \llbracket w \rrbracket^2 ds \right) \right]^{1/2}.$$

Let $f(\mathbf{u}, w)$ be a semi-norm that is continuous with respect to this norm such that there is a C only dependent on \mathcal{K} of \mathcal{T}_h and

$$|f(\mathbf{u}, w)| \leq C(\|\mathbf{u}\|_{\mathbf{H}_h^1} + \|w\|_{H_h^2}) \quad \forall (\mathbf{u}, w) \in \mathbf{H}_h^1 \times H_h^2.$$

We assume that f satisfies the rigid body motion condition that if $(\mathbf{u}, w) \in \mathbf{H}^1 \times H^2$ defines a rigid body motion and $f(\mathbf{u}, w) = 0$ then $\mathbf{u} = 0$ and $w = 0$. We have the discrete Korn's

inequality for Koiter shell that for all $\mathbf{u} \in \mathbf{H}_h^1$ and $w \in H_h^2$

$$(1.15) \quad \|\mathbf{u}\|_{\mathbf{H}_h^1} + \|w\|_{H_h^2} \leq C \left[\sum_{\alpha,\beta=1,2} \|\rho_{\alpha\beta}^K(\mathbf{u}, w)\|_{0,\Omega_h}^2 + \sum_{\alpha,\beta=1,2} \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{0,\Omega_h}^2 \right. \\ \left. + \sum_{e \in \mathcal{E}_h^0} \left(\sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket u_\alpha \rrbracket^2 ds + \sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket \partial_\alpha w \rrbracket^2 ds + h_e^{-1} \int_e \llbracket w \rrbracket^2 ds \right) + f^2(\mathbf{v}, z) \right]^{1/2}.$$

With this inequality established, one may add the term $\sum_{e \in \mathcal{E}_h^0} h_e^{-3} \int_e \llbracket w \rrbracket^2 ds$ to both sides to obtain a new inequality. Let $\llbracket \partial_s w \rrbracket$ and $\llbracket \partial_n w \rrbracket$ be the jumps in the tangential and normal derivatives of w over and edge e , respectively. Then we have the identity

$$\sum_{\alpha=1}^2 \llbracket \partial_\alpha w \rrbracket^2 = \llbracket \partial_s w \rrbracket^2 + \llbracket \partial_n w \rrbracket^2.$$

If w is a piecewise polynomial, we have the inverse inequality $\int_e \llbracket \partial_s w \rrbracket^2 \leq Ch_e^{-2} \int_e \llbracket w \rrbracket^2$. Thus for piecewise polynomials u_α and w , we have the following variant of discrete Korn's inequality for Koiter shell.

$$(1.16) \quad \|\mathbf{u}\|_{\mathbf{H}_h^1} + \|w\|_{\bar{H}_h^2} \leq C \left[\sum_{\alpha,\beta=1,2} \|\rho_{\alpha\beta}^K(\mathbf{u}, w)\|_{0,\Omega_h}^2 + \sum_{\alpha,\beta=1,2} \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{0,\Omega_h}^2 \right. \\ \left. + \sum_{e \in \mathcal{E}_h^0} \left(\sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket u_\alpha \rrbracket^2 ds + h_e^{-1} \int_e \llbracket \partial_n w \rrbracket^2 ds + h_e^{-3} \int_e \llbracket w \rrbracket^2 ds \right) + f^2(\mathbf{u}, w) \right]^{1/2}.$$

Here

$$(1.17) \quad \|w\|_{\bar{H}_h^2} := \left[\sum_{\tau \in \mathcal{T}_h} \|w\|_{2,\tau}^2 + \sum_{e \in \mathcal{E}_h^0} \left(h_e^{-1} \int_e \llbracket \partial_n w \rrbracket^2 ds + h_e^{-3} \int_e \llbracket w \rrbracket^2 ds \right) \right]^{1/2}.$$

This inequality is useful for analysis of discontinuous Galerkin methods for Koiter shell.

In proving these discrete Korn's inequalities, a important tool is a compact embedding theorem in the space H_h^1 , which is proved in Section 2. In Section 3 we prove the discrete Korn's inequalities (1.9) for the Naghdi shell, and Section 4 is devoted Koiter shell model. Throughout the paper, C will be a generic constant that may depend on the domain Ω , the mapping Φ that defines the shell mid-surface, and shape regularity \mathcal{K} of a triangle, of a triangulation, or of a class of triangulations. But otherwise, the constant is independent of triangulations. An integral $\int_\tau u(x_1, x_2) dx_1 dx_2$ or $\int_{\partial\Omega} v(s) ds$ will be simply written as $\int_\tau u$ or $\int_{\partial\Omega} v$, in which the integration variable and measure should be clear from the context.

2. COMPACT EMBEDDING IN THE SPACE OF PIECEWISE H^1 FUNCTIONS

We need a discrete analogue of the Rellich-Kondrachov compact embedding theorem, which will play a fundamental role in proving the discrete Korn's inequalities for shells. There are several papers relevant to compact embedding in piecewise function spaces. In [11], such a result is stated under the assumption that the triangulation \mathcal{T}_h is quasi-uniform. In [10], a theorem was proved for piecewise polynomials under the assumption that the maximum mesh size tends to zero. In [7], there is a sub-mesh condition to be verified. Although these theories are developed in more general settings, their results do not readily meet our needs. We only assume that \mathcal{T}_h is a shape regular triangulation of the polygon Ω . Our proof also clearly shows that the statement could break down when an interior angle of Ω tends to zero or 2π .

2.1. A trace theorem. We first prove a trace theorem for functions in H_h^1 . This result itself is a generalization of a trace theorem of Sobolev space theory. It will be used in proving the discrete compact embedding theorem, and in verifying the continuity of the seminorm denoted by f in the right hand side of (1.9), (1.15), and (1.16). We will need the following trace theorem on an element [3].

Lemma 2.1. *Let τ be a triangle, and e one of its edges. Then there is a constant C depending on the shape regularity of τ such that*

$$(2.1) \quad \int_e u^2 \leq C \left[h_e^{-1} \int_\tau u^2 + h_e \sum_{\alpha=1}^2 \int_\tau |\partial_\alpha u|^2 \right] \quad \forall u \in H^1(\tau).$$

Theorem 2.2. *Let \mathcal{T}_h be a shape regular, but not necessarily quasi-uniform triangulation of Ω . There exists a constant C such that*

$$(2.2) \quad \|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H_h^1} \quad \forall u \in H_h^1.$$

Proof. Let ϕ be a piecewise smooth vector field on Ω whose normal component is continuous across any straight line segment, and such that $\phi \cdot \mathbf{n} = 1$ on $\partial\Omega$. On each element $\tau \in \mathcal{T}_h$, we have

$$\int_{\partial\tau} u^2 \phi \cdot \mathbf{n} = \int_\tau \operatorname{div}(u^2 \phi) = \int_\tau (2u \nabla u \cdot \phi + u^2 \operatorname{div} \phi).$$

Here ∇u is the gradient of u . Summing up the above equations over all elements of \mathcal{T}_h , we get

$$\int_{\partial\Omega} u^2 = - \sum_{e \in \mathcal{E}_h^0} \int_e \llbracket u^2 \phi \rrbracket + \int_{\Omega_h} (2u \nabla u \cdot \phi + u^2 \operatorname{div} \phi).$$

If e is the border between the elements τ_1 and τ_2 with outward normals \mathbf{n}_1 and \mathbf{n}_2 , then $\llbracket u^2 \phi \rrbracket = u_1^2 \phi_1 \cdot \mathbf{n}_1 + u_2^2 \phi_2 \cdot \mathbf{n}_2$, where u_1 and u_2 are restrictions of u on τ_1 and τ_2 , respectively. It is noted that although ϕ may be discontinuous across e , its normal component is continuous,

i.e., $\phi_1 \cdot \mathbf{n}_1 + \phi_2 \cdot \mathbf{n}_2 = 0$. On the edge e , we have $|\llbracket u^2 \phi \rrbracket| \leq |\llbracket u^2 \rrbracket| \|\phi\|_{0,\infty,\Omega}$. Here, $|\llbracket u^2 \rrbracket| = |u_1^2 - u_2^2| = 2|\llbracket u \rrbracket \{u\}|$, with $\{u\} = (u_1 + u_2)/2$ being the average. We have

$$(2.3) \quad \int_e |\llbracket u^2 \phi \rrbracket| \leq 2\|\phi\|_{0,\infty,\Omega} \left[h_e^{-1} \int_e \llbracket u \rrbracket^2 \right]^{1/2} \left[h_e \int_e \{u\}^2 \right]^{1/2} \\ \leq C\|\phi\|_{0,\infty,\Omega} \left[h_e^{-1} \int_e \llbracket u \rrbracket^2 \right]^{1/2} \left[\int_{\delta e} u^2 + h_e^2 \int_{\delta e} |\nabla u|^2 \right]^{1/2}.$$

Here, $\|\phi\|_{0,\infty,\Omega}$ is the Sobolev norm in the space $[W^{0,\infty}(\Omega)]^2$ and C only depends on the shape regularity of τ_1 and τ_2 . We used $\delta e = \tau_1 \cup \tau_2$ to denote the “co-boundary” of edge e , and we used the trace estimate (2.1). It then follows from the Cauchy–Schwarz inequality that

$$\|u\|_{L^2(\partial\Omega)}^2 \leq C(\|\phi\|_{0,\infty,\Omega} + \|\operatorname{div} \phi\|_{0,\infty,\Omega}) \left[\|u\|_{L^2}^2 + \int_{\Omega_h} |\nabla u|^2 + \sum_{e \in \mathcal{E}_h^0} \frac{1}{|e|} \int_e \llbracket u \rrbracket^2 \right].$$

Here the constant C only depends on the shape regularity of \mathcal{T}_h . The dependence on Ω of the C in (2.2) is hidden in the ϕ in the above inequality. \square

2.2. Compact embedding in H_h^1 . For $\delta > 0$, we define a boundary strip Ω_δ of width $\mathcal{O}(\delta)$ for the domain Ω . Let $\Omega_\delta^0 = \Omega \setminus \overline{\Omega_\delta}$ be the interior domain. The interior domain Ω_δ^0 has the property that if a point is in Ω_δ^0 , then the disk centered at the point with radius δ entirely lies in Ω . We first show that when the strip is thin, the $L^2(\Omega_\delta)$ norm of the restriction on Ω_δ of a function in H_h^1 must be small.

Lemma 2.3. *There is a constant C such that*

$$(2.4) \quad \int_{\Omega_\delta} u^2 \leq C\delta \|u\|_{H_h^1}^2 \quad \forall u \in H_h^1.$$

Here Ω_δ is a boundary strip of width δ attached to $\partial\Omega$.

Proof. We choose a piecewise smooth a vector field ϕ whose normal component is continuous across any curve such that $\phi = 0$ on the inner boundary of Ω_δ , and $\operatorname{div} \phi = 1$ and $|\phi| \leq C\delta$ on Ω_δ . (A construction of such ϕ is given below.) We then extend ϕ by zero onto the entire domain Ω . The extended, still denoted by ϕ , is a piecewise smooth vector field whose normal components is continuous over any curve in Ω . We thus have

$$\int_{\Omega_\delta} u^2 = \int_{\Omega} u^2 \operatorname{div} \phi = \sum_{\tau \in \mathcal{T}_h} \int_{\tau} u^2 \operatorname{div} \phi = - \sum_{\tau \in \mathcal{T}_h} \int_{\tau} 2u \nabla u \cdot \phi + \sum_{\tau \in \mathcal{T}_h} \int_{\partial\tau} u^2 \phi \cdot \mathbf{n}.$$

The last term can be written as

$$\sum_{\tau \in \mathcal{T}_h} \int_{\partial\tau} u^2 \phi \cdot \mathbf{n} = \int_{\partial\Omega} u^2 \phi \cdot \mathbf{n} + \sum_{e \in \mathcal{E}_h^0} \int_e \llbracket u^2 \phi \rrbracket.$$

Since the normal components of ϕ is continuous on edges in \mathcal{E}_h^0 , we use the same argument as in the proof of Theorem 2.2, cf., (2.3), to get

$$\sum_{e \in \mathcal{E}_h^0} \int_e |\llbracket u^2 \phi \rrbracket| \leq C |\phi|_{0,\infty,\Omega} \left[\|u\|_{L^2}^2 + \sum_{\tau \in \mathcal{T}_h} \int_{\tau} h_{\tau}^2 |\nabla u|^2 + \sum_{e \in \mathcal{E}_h^0} \frac{1}{h_e} \int_e \llbracket u \rrbracket^2 \right].$$

It then follows from Theorem 2.2 that

$$(2.5) \quad \int_{\Omega_{\delta}} u^2 \leq C |\phi|_{0,\infty,\Omega} \left[\|u\|_{L^2}^2 + \int_{\Omega_h} |\nabla u|^2 + \sum_{e \in \mathcal{E}_h^0} \frac{1}{h_e} \int_e \llbracket u \rrbracket^2 \right].$$

The proof is complete since $|\phi|_{0,\infty,\Omega} \leq C\delta$. The constant C depends on Ω in terms of its interior angles and exterior angles at convex and concave vertexes, respectively. \square

We describe a way to choose the boundary strip and construct the vector field ϕ that was used in the proof. This field can be constructed by piecing together several special vector fields. We need some vector fields on rectangles, wedges, and circular disks. On the xy -plane, we consider the vertical rectangular strip $R = (0, \delta) \times (0, l)$. On this strip, we consider $\psi_R = \langle x, 0 \rangle$. This vector field satisfies the condition that $\operatorname{div} \psi_R = 1$, $\psi_R = 0$ on the left side, $\psi_R \cdot \mathbf{n} = 0$ on the top and bottom sides and the maximum of $|\psi_R|$ is δ that is attained on the right side. On a wedge W with its vertex at the origin, we consider the vector field $\psi_W = \langle x, y \rangle / 2$. This ψ_W satisfies the conditions that $\operatorname{div} \psi_W = 1$, $\psi_W \cdot \mathbf{n} = 0$ on the two sides of W , and $|\psi_W| = \rho/2$ at a point in W whose distant from the origin is ρ . On a circular disk C centered at the origin and of radius ρ , we consider the vector field $\psi_C = (1 - \rho^2/r^2) \langle x, y \rangle / 2$. Here $r = (x^2 + y^2)^{1/2}$. This vector field satisfies the condition that $\operatorname{div} \psi_C = 1$ on the disk except at the center where it is singular. It points toward the center every where. And it is zero on the boundary. We use this field on a sector of the circle C , on the two radial sides of which we have $\psi_C \cdot \mathbf{n} = 0$. With these special vector

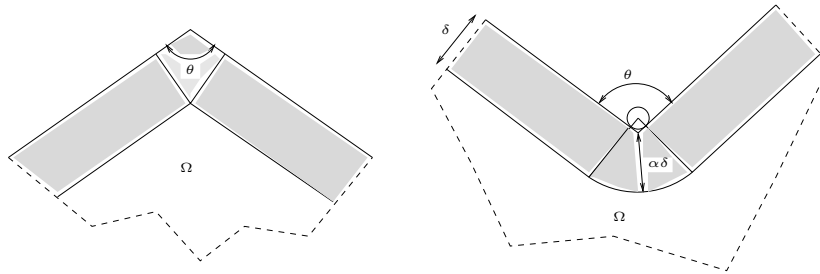


FIGURE 1. Boundary strip Ω_δ near a convex vertex (left) and a concave vertex (right).

fields, we can then assemble the ϕ on a boundary strip Ω_δ . Along the interior side of each straight segment of $\partial\Omega$ we choose a uniform strip of thickness δ . These strips overlap near vertexes. If Ω is convex at a vertex, we introduce a wedge whose vertex is at the intersection

of the interior boundary of the uniform strips, and whose sides are orthogonal to the meeting straight segments, see the left figure of Figure 1. If Ω is concave at a vertex, we resolve it by using a circular sector, centered near the vertex and outside of Ω . The radius of the circle is slightly bigger than δ such that the arc is continuously connected to the interior edges of the meeting strips, and the two radial sides are orthogonal to the meeting boundary segments, see the right figure in Figure 1. With such treatment of the vertices, the boundary strip Ω_δ is composed of rectangular strips attaching to major portions of straight segments of $\partial\Omega$, portion of wedges at convex vertexes, and portion of circular sectors at concave vertexes, see the shaded region in Figure 1. We then transform ψ_R , ψ_W , and ψ_C to various parts of Ω_δ and assemble a ϕ on Ω_δ . The vector field ϕ thus constructed is zero on the interior boundary of Ω_δ . Its normal components are continuous across any curve, and $\operatorname{div} \phi = 1$ on Ω_δ . The thickness of Ω_δ is the constant δ for the rectangular part. It is maximized to $\delta/\sin \frac{\theta}{2}$ at a convex vertex. It is minimized to $\alpha\delta$ at the concave vertex, with $0 < \alpha < 1$, a value can be chosen as, for example, $1/2$. The norm $|\phi|$ has a maximum $\delta/\sin \frac{\theta}{2}$ at a convex vertex with θ being the interior angle. Thus when θ is small, $|\phi|$ is big, and the estimate (2.5) deteriorates. The norm $|\phi|$ also has a local maximum near a concave vertex. It is bounded as

$$|\phi| \leq \delta \frac{1 - \alpha \sin \frac{\theta}{2}}{(1 - \alpha) \sin \frac{\theta}{2}}.$$

When the exterior angle is small this maximum would be big. Also, one needs to choose α away from 1 and 0, to maintain a moderate thickness of the strip and a reasonable bound for $|\phi|$ which affect the estimate (2.5). We remark that the constant C in the estimate (2.4) could tend to infinity if an interior angle of the polygonal domain Ω tends to zero or 2π .

We then prove that functions in H_h^1 are “shift-continuous” in L^2 , as stated in the following lemma. We extend a function $u \in H_h^1$ to a function \tilde{u} on the whole \mathbb{R}^2 by zero.

Lemma 2.4. *There is a constant C such that*

$$(2.6) \quad \int_{\mathbb{R}^2} [\tilde{u}(\mathbf{x} + \boldsymbol{\rho}) - \tilde{u}(\mathbf{x})]^2 d\mathbf{x} \leq C|\boldsymbol{\rho}| \|u\|_{H_h^1}^2 \quad \forall u \in H_h^1.$$

Proof. Because for an element $\tau \in \mathcal{T}_h$, smooth functions are dense in $H^1(\tau)$, we only need to prove (2.6) for functions that are smooth on each element of \mathcal{T}_h . Let u be such a piecewise smooth function. Let $\boldsymbol{\rho}$ be an arbitrary short vector. We take $\delta = |\boldsymbol{\rho}|$ and choose a boundary strip Ω_δ . The interior part Ω_δ^0 of the domain has the property that if $\mathbf{x} \in \Omega_\delta^0$ then the line segment $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \subset \Omega$. We have

$$\int_{\mathbb{R}^2} [\tilde{u}(\mathbf{x} + \boldsymbol{\rho}) - \tilde{u}(\mathbf{x})]^2 d\mathbf{x} = \int_{\Omega_\delta^0} [u(\mathbf{x} + \boldsymbol{\rho}) - u(\mathbf{x})]^2 d\mathbf{x} + \int_{\mathbb{R}^2 \setminus \Omega_\delta^0} [\tilde{u}(\mathbf{x} + \boldsymbol{\rho}) - \tilde{u}(\mathbf{x})]^2 d\mathbf{x}.$$

Using Lemma 2.3, we bound the second term as

$$(2.7) \quad \int_{\mathbb{R}^2 \setminus \Omega_\delta^0} [\tilde{u}(\mathbf{x} + \boldsymbol{\rho}) - \tilde{u}(\mathbf{x})]^2 d\mathbf{x} \leq \int_{\Omega_{2\delta}} u^2(\mathbf{x}) d\mathbf{x} \leq C|\boldsymbol{\rho}| \|u\|_{H_h^1}^2.$$

We then focus on the first term. This integral can be taken on an equal measure subset of Ω_δ^0 . This subset is obtained by removing a zero measure subset that is composed of such point \mathbf{x} : \mathbf{x} or $\mathbf{x} + \boldsymbol{\rho}$ is on an open edge $e \in \mathcal{E}_h^0$, or the closed straight line segment $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$ connecting \mathbf{x} and $\mathbf{x} + \boldsymbol{\rho}$ contains any vertex of \mathcal{T}_h , or $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$ overlaps some edges of \mathcal{E}_h^0 . By such exclusion, for any \mathbf{x} in the remaining set, both the ends of $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$ are in the interior of some open triangular elements, and $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$ contains no vertex. The restriction of u on $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$ is a piecewise smooth one dimensional function, which may have a finite number of jumping points in the open straight line segment $(\mathbf{x}, \mathbf{x} + \boldsymbol{\rho})$. By the fundamental theorem of calculus, we have

$$u(\mathbf{x} + \boldsymbol{\rho}) - u(\mathbf{x}) = \int_0^1 \nabla u(\mathbf{x} + t\boldsymbol{\rho}) \cdot \boldsymbol{\rho} dt + \sum_{p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0} \llbracket u \rrbracket_p$$

Note that the integrand in the integral may make no sense at t , if $\mathbf{x} + t\boldsymbol{\rho} \in \mathcal{E}_h^0$, where u may jump. These points are excluded from the integration, where the jumping effect is resolved in the second term. On the segment $[\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]$, u may have a jump at $p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0$, which is denoted by $\llbracket u \rrbracket_p$ that is the value of u from the side of \mathbf{x} minus that from the side of $\mathbf{x} + \boldsymbol{\rho}$. We thus have

$$[u(\mathbf{x} + \boldsymbol{\rho}) - u(\mathbf{x})]^2 \leq |\boldsymbol{\rho}|^2 \int_0^1 |\nabla u(\mathbf{x} + t\boldsymbol{\rho})|^2 dt + \left[\sum_{p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0} \llbracket u \rrbracket_p \right]^2.$$

When we take integral on Ω_δ^0 (minus the aforementioned zero-measure subset), the first term is bounded as follows.

$$\int_{\Omega_\delta^0} |\boldsymbol{\rho}|^2 \int_0^1 |\nabla u(\mathbf{x} + t\boldsymbol{\rho})|^2 dt d\mathbf{x} = |\boldsymbol{\rho}|^2 \int_0^1 \int_{\Omega_\delta^0} |\nabla u(\mathbf{x} + t\boldsymbol{\rho})|^2 d\mathbf{x} dt \leq |\boldsymbol{\rho}|^2 \int_{\Omega_h} |\nabla u|^2 d\mathbf{x}.$$

To estimate the jumping related second term, we write $\llbracket u \rrbracket_p = h_e^{1/2} h_e^{-1/2} \llbracket u \rrbracket_p$ if $p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0$ is on the edge e , and use the Cauchy-Schwarz inequality to obtain the following estimate.

$$\left[\sum_{p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0} \llbracket u \rrbracket_p \right]^2 \leq \left[\sum_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \neq \emptyset} h_e^{-1} \llbracket u \rrbracket_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]}^2 \right] \left[\sum_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \neq \emptyset} h_e \right].$$

We show below that there is a C , depending on the domain Ω and the shape regularity \mathcal{K} of \mathcal{T}_h , such that

$$(2.8) \quad \sum_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \neq \emptyset} h_e \leq C.$$

We then have

$$\int_{\Omega_\delta^0} \left[\sum_{p \in [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \cap \mathcal{E}_h^0} \llbracket u \rrbracket_p \right]^2 \leq C \int_{\Omega_\delta^0} \sum_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \neq \emptyset} h_e^{-1} \llbracket u \rrbracket_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]}^2 d\mathbf{x}.$$

Every term in the right hand side is associated with a particular edge $e \in \mathcal{E}_h^0$. Each edge $e \in \mathcal{E}_h^0$ is relevant to at most the points in the parallelogram Ω_e in the Figure 2. Thus, by changing the order of sum and integral, we have

$$\begin{aligned} \int_{\Omega_\delta^0} \sum_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}] \neq \emptyset} h_e^{-1} \llbracket u \rrbracket_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]}^2 d\mathbf{x} &\leq \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \int_{\Omega_e} \llbracket u \rrbracket_{e \cap [\mathbf{x}, \mathbf{x} + \boldsymbol{\rho}]}^2 d\mathbf{x} \\ &\leq \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \sin \langle \boldsymbol{\rho}, e \rangle |\boldsymbol{\rho}| \int_e \llbracket u \rrbracket^2 \leq |\boldsymbol{\rho}| \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \int_e \llbracket u \rrbracket^2. \end{aligned}$$

Here, $\langle \boldsymbol{\rho}, e \rangle$ is the angle between the vector $\boldsymbol{\rho}$ and the edge e . Therefore, we have

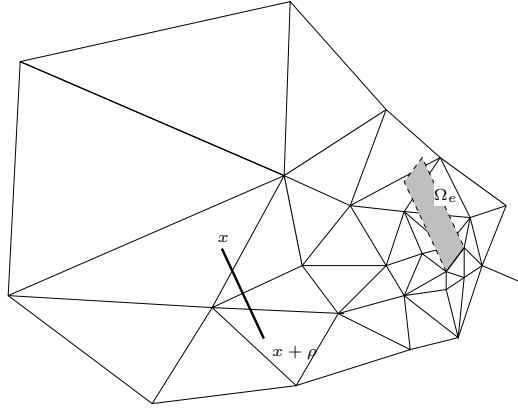


FIGURE 2. A $\boldsymbol{\rho}$ shift and Ω_e for an edge e .

$$\int_{\Omega_\delta^0} [u(\mathbf{x} + \boldsymbol{\rho}) - u(\mathbf{x})]^2 d\mathbf{x} \leq |\boldsymbol{\rho}|^2 |\nabla u|_{0, \Omega_h}^2 + |\boldsymbol{\rho}| \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \int_e \llbracket u \rrbracket^2.$$

Note that the second term may carry the smaller coefficient $|\boldsymbol{\rho}| \max\{h, |\boldsymbol{\rho}|\}$ such that the two terms are closer in the order. But we do not need such refined estimates. We thus proved

$$(2.9) \quad \int_{\Omega_\delta^0} [u(\mathbf{x} + \boldsymbol{\rho}) - u(\mathbf{x})]^2 d\mathbf{x} \leq C |\boldsymbol{\rho}| \|u\|_{H_h^1}^2 \quad \forall u \in H_h^1.$$

The shift continuity (2.6) then follows from (2.9) and (2.7). We have shown that the set of zero extended functions is shift continuous in $L^2(\mathbb{R}^2)$. \square

We give a proof for the estimate (2.8). Let l be a straight line cutting through Ω . Let \mathcal{T}_h be a shape regular triangulation with regularity constant \mathcal{K} . Then the sum of lengths of mesh line segments intersecting l is bounded independent of the triangulation. More specifically, we prove that there is a constant C , depending on the shape regularity \mathcal{K} , but otherwise independent of the triangulation \mathcal{T}_h such that

$$(2.10) \quad \sum_{e \in \mathcal{E}_h \text{ and } e \cap l \neq \emptyset} h_e \leq C.$$

We shall use some facts that follow from the shape regularity assumption. There is a minimum angle $\theta_{\mathcal{K}}$ among all angles of triangles of \mathcal{T}_h . The number of edges sharing a vertex is bounded by a constant C that only depends on \mathcal{K} . Let e_1 and e_2 be two edges sharing a vertex. There are constants C_1 and C_2 depending on \mathcal{K} such that $h_{e_1} \leq C_1 h_{e_2}$ and $h_{e_2} \leq C_2 h_{e_1}$. Without loss of generality, we assume l is horizontal. We also assume that l does not pass any vertex. (This assumption was already made in the context of (2.8). It can be removed by a slight modification of the following argument.) We first trim the set of intersecting

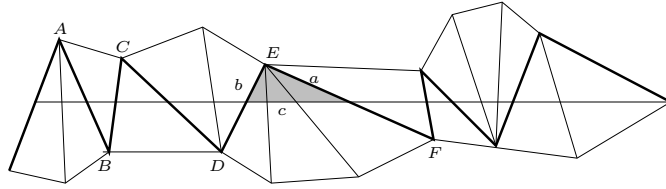


FIGURE 3. A line cutting through the triangulation.

edges $\{e \in \mathcal{E}_h \text{ and } e \cap l \neq \emptyset\}$ to simplify the set without significantly reducing its sum of lengths of all edges. Consider the left most edge intersecting l , of which only one end vertex is shared by some other edges intersecting l . Let A be this end vertex, and assume it is above l . We examine all the edges intersecting l and sharing A in the counterclockwise order. We discard all such edges but the last one that is AB in Figure 3. (The next edge sharing A , as AC , does not intersect l .) The edge BC must intersect l . There could be other edges intersecting l and sharing the vertex B . We examine all the edges sharing B and intersecting l in the clockwise order. We discard all but the last one. (It is BC in the figure.) Now the vertex C is in the same situation as A , and we can determine the edge CD using the same rule as for AB . Then we determine DE , EF , and so forth. We do the trimming all the way to the right end of l . This procedure touches every edge intersecting l , by either trimming an edge off or keeping it. Every edge deleted has at least one vertex-sharing edge retained. The remaining edges constitute a continuous piecewise straight path as represented by the

thick line in the figure. We denote this set by \mathcal{E}_h^l . It follows from the aforementioned facts about the shape regular triangulation that there is a constant C , depending on \mathcal{K} only, such that

$$\sum_{e \in \mathcal{E}_h \text{ and } e \cap l \neq \emptyset} h_e \leq C \sum_{e \in \mathcal{E}_h^l} h_e.$$

We consider a typical triangle bounded by l and \mathcal{E}_h^l , as the shaded one in the figure, whose sides are a , b , and c . Let the angle $\angle DEF$ be denoted by θ . Then $\theta \geq \theta_{\mathcal{K}}$. Note that $c^2 = a^2 + b^2 - 2ab \cos \theta$. If θ is obtuse, then $a + b \leq \sqrt{2}c$. Otherwise, we have $c^2 = (a^2 + b^2)(1 - \cos \theta) + (a - b)^2 \cos \theta \leq (a^2 + b^2)(1 - \cos \theta)$. Thus $a + b \leq \sqrt{\frac{2}{1 - \cos \theta}}c$. In any case, we have $a + b \leq \sqrt{\frac{2}{1 - \cos \theta_{\mathcal{K}}}}c$. We thus proved

$$\sum_{e \in \mathcal{E}_h^l \cap \mathcal{E}_h^0} h_e \leq \sqrt{\frac{2}{1 - \cos \theta_{\mathcal{K}}}} |l \cap \Omega|.$$

From this, (2.10) follows. Here $|l \cap \Omega|$ is the length of the line segment $l \cap \Omega$ which does exceed the diameter of Ω . We can now prove the following compact embedding theorem.

Theorem 2.5. *Let \mathcal{T}_{h_i} be a (infinite) class of shape regular but not necessarily quasi-uniform triangulations of the polygonal domain Ω , with a shape regularity constant \mathcal{K} . For each i , let $H_{h_i}^1$ be the space of piecewise H^1 functions, subordinated to the triangulation T_{h_i} , equipped with the norm (1.8). Let $\{u_i\}$ be a sequence such that $u_i \in H_{h_i}^1$ for each i and there is a constant C , such that $\|u_i\|_{H_{h_i}^1} \leq C$ for all i . Then, the sequence $\{u_i\}$ has a convergent subsequence in L^2 .*

Proof. It follows from (2.6) that the sequence $\{u_i\}$ is a shift-continuous subset of L^2 . This, together with the estimate (2.2), verifies the condition for a subset of L^2 to be compact, see Theorem 2.12 in [1]. \square

3. DISCRETE KORN'S INEQUALITY FOR NAGHDI SHELL

In this section, we prove the discrete Korn's inequality (1.9) for the Naghdi shell model. We let $H_h = \mathbf{H}_h^1 \times \mathbf{H}_h^1 \times H_h^1$, and define a norm in this space by

$$(3.1) \quad \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h} := \left[\sum_{\alpha=1,2} \left(\|\theta_\alpha\|_{H_h^1}^2 + \|u_\alpha\|_{H_h^1}^2 \right) + \|w\|_{H_h^1}^2 \right]^{1/2}.$$

Recall that the norm in H_h^1 is defined by (1.8). Let $f(\boldsymbol{\theta}, \mathbf{u}, w)$ be a semi-norm that is continuous with respect to the H_h norm such that there is a C and

$$(3.2) \quad |f(\boldsymbol{\theta}, \mathbf{u}, w)| \leq C \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h} \quad \forall (\boldsymbol{\theta}, \mathbf{u}, w) \in H_h.$$

We also assume that f satisfies the condition that if $(\boldsymbol{\theta}, \mathbf{u}, w) \in \mathbf{H}^1 \times \mathbf{H}^1 \times H^1$ defines a rigid body motion, as explained in the introduction, and $f(\boldsymbol{\theta}, \mathbf{u}, w) = 0$ then $\boldsymbol{\theta} = 0$, $\mathbf{u} = 0$, and $w = 0$.

We define a discrete energy norm on the space H_h by

$$(3.3) \quad \begin{aligned} \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{E_h} := & \left[\sum_{\alpha, \beta=1,2} (\|\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w)\|_{0,\Omega_h}^2 + \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{0,\Omega_h}^2) + \sum_{\alpha=1,2} \|\tau_{\alpha}(\boldsymbol{\theta}, \mathbf{u}, w)\|_{0,\Omega_h}^2 \right. \\ & \left. + \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \left(\sum_{\alpha=1,2} \int_e ([\theta_{\alpha}]^2 + [u_{\alpha}]^2) + \int_e [w]^2 \right) + f^2(\boldsymbol{\theta}, \mathbf{u}, w) \right]^{1/2}. \end{aligned}$$

The main result of this section is the following theorem.

Theorem 3.1. *We assume that $\Omega \subset \mathbb{R}^2$ is a polygon, on which \mathcal{T}_h is a shape regular but not necessarily quasi-uniform triangulation with a regularity constant \mathcal{K} . Let $f(\boldsymbol{\theta}, \mathbf{u}, w)$ be a seminorm on H_h satisfies the condition (3.2), and the rigid body motion condition. There exists a constant C that could be dependent on the shell mid-surface and shape regularity of the triangulation, but otherwise independent of the triangulation, such that*

$$(3.4) \quad \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h} \leq C \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{E_h} \quad \forall (\boldsymbol{\theta}, \mathbf{u}, w) \in H_h.$$

To prove this theorem, we need a discrete Korn's inequality for plane elasticity for piecewise functions in H_h^1 , see (1.21) of [6]. It says that there is a constant C that might be dependent on the domain Ω and the shape regularity \mathcal{K} of the triangulation \mathcal{T}_h , but otherwise independent of \mathcal{T}_h such that

$$(3.5) \quad \sum_{\alpha=1,2} \|u_{\alpha}\|_{H_h^1}^2 \leq C \left[\sum_{\alpha=1,2} \|u_{\alpha}\|_{0,\Omega_h}^2 + \sum_{\alpha,\beta=1,2} \|e_{\alpha\beta}(\mathbf{u})\|_{0,\Omega_h}^2 + \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \int_e \sum_{\alpha=1,2} [u_{\alpha}]^2 \right].$$

Here $e_{\alpha\beta}(\mathbf{u}) = (\partial_{\beta}u_{\alpha} + \partial_{\alpha}u_{\beta})/2$ is the plane elasticity strain which is the symmetric part of the gradient of \mathbf{u} .

Proof of Theorem 3.1. From (3.5), the definitions (1.1), (1.2), and (1.3) of $\rho_{\alpha\beta}$, $\gamma_{\alpha\beta}$, and τ_{α} , and the definition of the discrete energy norm (3.3), we see that there is a constant C such that

$$(3.6) \quad \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h}^2 \leq C \left[\|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{E_h}^2 + \sum_{\alpha=1,2} (\|\theta_{\alpha}\|_{0,\Omega_h}^2 + \|u_{\alpha}\|_{0,\Omega_h}^2) + \|w\|_{0,\Omega_h}^2 \right] \\ \forall (\boldsymbol{\theta}, \mathbf{u}, w) \in H_h.$$

On a fixed triangulation \mathcal{T}_h , it then follows from the Rellich–Kondrachov compact embedding theorem and Peetre’s lemma (Theorem 2.1, page 18 in [12]) that there is a constant $C_{\mathcal{T}_h}$ such that

$$\|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h} \leq C_{\mathcal{T}_h} \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{E_h} \quad \forall (\boldsymbol{\theta}, \mathbf{u}, w) \in H_h.$$

We need to show that for a class of shape regular triangulations, such $C_{\mathcal{T}_h}$ has an upper bound that only depends on the shape regularity \mathcal{K} of the whole class. If this is not true, there would exist a sequence of triangulations $\{\mathcal{T}_{h_n}\}$ and an associated sequence of functions $(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)$ in H_{h_n} such that

$$(3.7) \quad \|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{H_{h_n}} = 1 \text{ and } \|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{E_{h_n}} \leq 1/n.$$

It follows from Theorem 2.5 that there is a subsequence, still denoted by $(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)$, such that

$$(3.8) \quad \lim_{n \rightarrow \infty} (\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) = (\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) \text{ in } \mathbf{L}^2 \times \mathbf{L}^2 \times L^2.$$

We show that the limiting functions $\boldsymbol{\theta}^0$, \mathbf{u}^0 , and w^0 are all in H^1 , this limit defines a rigid body motion, and it is zero, which will lead to a contradiction.

First, we show that w^0 is actually in H^1 and we have that $\partial_\alpha w^0 + \theta_\alpha^0 + b_\alpha^\beta u_\beta^0 = 0$. Since $\lim_{n \rightarrow \infty} (\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) = (\boldsymbol{\theta}^0, \mathbf{u}^0, w^0)$ in $\mathbf{L}^2 \times \mathbf{L}^2 \times L^2$, in view of the definition (1.3), we have

$$\lim_{n \rightarrow \infty} \tau_\alpha(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) = \tau_\alpha(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) \text{ in } H^{-1}.$$

In the above expressions, the derivatives $\partial_\alpha w^n$ and $\partial_\alpha w^0$ are understood in distributional sense.

For any compactly supported smooth functions $\phi^\alpha \in \mathcal{D}(\Omega)$, for each n , we have

$$\begin{aligned} \langle \tau_\alpha(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n), \phi^\alpha \rangle &= - \int_{\Omega_{h_n}} w^n \partial_\alpha \phi^\alpha + \int_{\Omega_{h_n}} (\theta_\alpha^n + b_\alpha^\beta u_\beta^n) \phi^\alpha \\ &= \int_{\Omega_{h_n}} (\partial_\alpha w^n + \theta_\alpha^n + b_\alpha^\beta u_\beta^n) \phi^\alpha - \sum_{e \in \mathcal{E}_{h_n}^0} \int_e \llbracket w^n \rrbracket_{n_\alpha} \phi^\alpha. \end{aligned}$$

Here, n_α are the components of the unit normal \mathbf{n} to the edge e . If e is shared by τ_1 and τ_2 with unit outward normals being \mathbf{n}_1 and \mathbf{n}_2 , on which the restrictions of w are w_1 and w_2 , then $\llbracket w \rrbracket_{n_\alpha} = w_1 n_{1\alpha} + w_2 n_{2\alpha}$ is the jump of w with respect to \mathbf{n} . Summation convention is also used in $\llbracket w^n \rrbracket_{n_\alpha} \phi^\alpha$ with α being viewed as a repeated sub and super scripts. Using Hölder inequality and the trace inequality (2.1), we get

$$\begin{aligned} |\langle \tau_\alpha(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n), \phi^\alpha \rangle| &\leq \|\tau_\alpha(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{0, \Omega_{h_n}} \|\phi^\alpha\|_{0, \Omega} \\ &\quad + C \left[\sum_{e \in \mathcal{E}_{h_n}^0} h_e^{-1} \int_e \llbracket w^n \rrbracket^2 \right]^{1/2} \sum_{\alpha=1}^2 \left[|\phi^\alpha|_{0, \Omega}^2 + \sum_{\tau \in \mathcal{T}_{h_n}} h_\tau^2 |\phi^\alpha|_{1, \tau}^2 \right]^{1/2}. \end{aligned}$$

Since $\|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{E_{h_n}} \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \tau_\alpha(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) = 0 \text{ in } H^{-1}.$$

Therefore, in H^{-1} , we have $\tau_\alpha(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = \partial_\alpha w^0 + \theta_\alpha^0 + b_\alpha^\beta u_\beta^0 = 0$. Since θ_α^0 and u_α^0 are in L^2 , this equation shows that the weak derivatives of w^0 are in L^2 . Thus we have $w^0 \in H^1$ and

$$(3.9) \quad \tau_\alpha(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = \partial_\alpha w^0 + \theta_\alpha^0 + b_\alpha^\beta u_\beta^0 = 0.$$

Next we show $u_\alpha^0 \in H^1$ and $\gamma_{\alpha\beta}(\mathbf{u}^0, w^0) = 0$. We have $\lim_{n \rightarrow \infty} \gamma_{\alpha\beta}(\mathbf{u}^n, w^n) = \gamma_{\alpha\beta}(\mathbf{u}^0, w^0)$ in H^{-1} , in which the derivatives $\partial_\alpha u_\beta^n$ and $\partial_\alpha u_\beta^0$ are understood in the distributional sense. Let $\phi^{\alpha\beta}$ be an arbitrary symmetric tensor valued function with components in $\mathcal{D}(\Omega)$. For each n we have

$$\begin{aligned} \langle \gamma_{\alpha\beta}(\mathbf{u}^n, w^n), \phi^{\alpha\beta} \rangle &= - \int_\Omega u_\alpha^n \partial_\beta \phi^{\alpha\beta} - \int_\Omega (\Gamma_{\alpha\beta}^\lambda u_\lambda^n + b_{\alpha\beta} w^n) \phi^{\alpha\beta} \\ &= \int_{\Omega_{h_n}} \gamma_{\alpha\beta}(\mathbf{u}^n, w^n) \phi^{\alpha\beta} - \sum_{e \in \mathcal{E}_{h_n}^0} \int_e \llbracket u_\alpha^n \rrbracket_{n_\beta} \phi^{\alpha\beta}. \end{aligned}$$

It follows from this equation and the assumption $\lim_{n \rightarrow \infty} \|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{E_{h_n}} = 0$ that we have $\lim_{n \rightarrow \infty} \langle \gamma_{\alpha\beta}(\mathbf{u}^n, w^n), \phi^{\alpha\beta} \rangle = 0$. Thus $\gamma_{\alpha\beta}(\mathbf{u}^0, w^0) = 0$, in which the derivatives are understood in the distributional sense. In view of the definition (1.2), we have

$$e_{\alpha\beta}(\mathbf{u}^0) = \Gamma_{\alpha\beta}^\lambda u_\lambda^0 + b_{\alpha\beta} w^0.$$

Since u_α^0 and w^0 are in L^2 , so $e_{\alpha\beta}(\mathbf{u}^0) \in L^2$. In the sense of distribution, there is the identity that [5, 14]

$$\partial_{\alpha\beta} u_\lambda^0 = \partial_\beta e_{\alpha\lambda}(\mathbf{u}^0) + \partial_\alpha e_{\lambda\beta}(\mathbf{u}^0) - \partial_\lambda e_{\alpha\beta}(\mathbf{u}^0).$$

From this we see that $\partial_{\alpha\beta} u_\lambda^0 \in H^{-1}$. It follows from a Lemma of J.L. Lions (whose assumption on the domain is met by our polygon, see page 110 of [14] and page 124 of [5]) and the fact $\partial_\beta u_\lambda^0 \in H^{-1}$ that $\partial_\beta u_\lambda^0 \in L^2$. Therefore, we proved that $u_\alpha^0 \in H^1$, and we have

$$(3.10) \quad \gamma_{\alpha\beta}(\mathbf{u}^0, w^0) = 0.$$

We then show that $\theta_\alpha^0 \in H^1$ and $\rho_{\alpha\beta}(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0$. As in the above, in the space H^{-1} , we have $\lim_{n \rightarrow \infty} \rho_{\alpha\beta}(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) = \rho_{\alpha\beta}(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0)$, in which the derivatives are understood in the distributional sense. For arbitrary $\phi^{\alpha\beta} \in \mathcal{D}(\Omega)$, and for any n , we have the identity

$$\langle \rho_{\alpha\beta}(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n), \phi^{\alpha\beta} \rangle = \int_{\Omega_{h_n}} \rho_{\alpha\beta}(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) \phi^{\alpha\beta} - \sum_{e \in \mathcal{E}_{h_n}^0} \int_e \llbracket \theta_\alpha^n \rrbracket_{n_\beta} \phi^{\alpha\beta} + \sum_{e \in \mathcal{E}_{h_n}^0} \int_e b_\alpha^\gamma \llbracket u_\gamma^n \rrbracket_{n_\beta} \phi^{\alpha\beta}.$$

Using the assumption $\lim_{n \rightarrow \infty} \|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{E_{h_n}} \rightarrow 0$ and this equation, we get

$$\lim_{n \rightarrow \infty} \langle \rho_{\alpha\beta}(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n), \phi^{\alpha\beta} \rangle = 0.$$

Therefore $\rho_{\alpha\beta}(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0$, in which the derivatives $\partial_\alpha \theta_\beta^0$ are understood in the distributional sense. In view of the definition (1.1) of $\rho_{\alpha\beta}(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0)$, and the proved regularity that $u_\alpha^0 \in H^1$, we see that $e_{\alpha\beta}(\boldsymbol{\theta}^0)$ are in L^2 . From this and the argument used above, we see $\partial_{\alpha\beta}^2 \theta_\gamma \in H^{-1}$. Using the Lemma of J. L. Lions again, we get the regularity $\theta_\alpha^0 \in H^1$, and the equation

$$(3.11) \quad \rho_{\alpha\beta}(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0.$$

According to Lemma 3.4 of [5], using the regularities that θ_α^0 , u_α^0 , and w^0 are all in H^1 , and the equations (3.9), (3.10), and (3.11), we conclude that the displacement functions $(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0)$ defines a rigid body motion.

Finally, we show that $(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0$. It follows from the bound (3.6), the assumption (3.7), and the convergence (3.8) that $\lim_{n \rightarrow \infty} \|(\boldsymbol{\theta}^n - \boldsymbol{\theta}^0, \mathbf{u}^n - \mathbf{u}^0, w^n - w^0)\|_{H_{h_n}} = 0$. Since f is uniformly continuous with respect to the norm of H_{h_n} and since $f(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n) \rightarrow 0$ (f is a part of the energy norm (3.3)), we see $f(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0$. Thus $(\boldsymbol{\theta}^0, \mathbf{u}^0, w^0) = 0$. Therefore, $\lim_{n \rightarrow \infty} \|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{H_{h_n}} = 0$, which is contradict to the assumption that $\|(\boldsymbol{\theta}^n, \mathbf{u}^n, w^n)\|_{H_{h_n}} = 1$. \square

As an example, we take

$$f(\boldsymbol{\theta}, \mathbf{u}, w) = \left[\sum_{e \in \mathcal{E}_h^D} \int_e \sum_{\alpha=1,2} \theta_\alpha^2 + \sum_{e \in \mathcal{E}_h^S \cup \mathcal{E}_h^D} \left(\int_e \sum_{\alpha=1,2} u_\alpha^2 + \int_e w^2 \right) \right]^{1/2}.$$

It follows from Theorem 2.2 that there is a C such that the continuity condition (3.2) is satisfied by this f . Under the assumption that the measure of $\partial^D \Omega$ is positive, it is verified in [5] that if $(\boldsymbol{\theta}, \mathbf{u}, w)$ defines a rigid body motion and $f(\boldsymbol{\theta}, \mathbf{u}, w) = 0$ then $\boldsymbol{\theta} = 0$, $\mathbf{u} = 0$, and $w = 0$. With this f in the Korn's inequality (3.4), we could add boundary penalty term

$$\sum_{e \in \mathcal{E}_h^D} \int_e h_e^{-1} \sum_{\alpha=1,2} \theta_\alpha^2 + \sum_{e \in \mathcal{E}_h^S \cup \mathcal{E}_h^D} \left(h_e^{-1} \int_e \sum_{\alpha=1,2} u_\alpha^2 + h_e^{-1} \int_e w^2 \right)$$

to the squares of both sides of (3.4) to obtain an inequality that is useful in analysis of discontinuous Galerkin methods for Naghdi shell in which the essential boundary conditions are enforced by Nitsche's consistent boundary penalty method.

4. DISCRETE KORN'S INEQUALITY FOR KOITER SHELL

The wellposedness of the Koiter model (1.12) is based on the Korn's inequality for Koiter shell that there is a constant C such that

$$(4.1) \quad \sum_{\alpha=1,2} \|u_\alpha\|_{H^1}^2 + \|w\|_{H^2}^2 \leq C \left[\sum_{\alpha,\beta=1,2} \|\rho_{\alpha\beta}^K(\mathbf{u}, w)\|_{L^2}^2 + \sum_{\alpha,\beta=1,2} \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{L^2}^2 + f^2(\mathbf{u}, w) \right] \\ \forall \mathbf{u} \in \mathbf{H}^1, w \in H^2.$$

Here $f(\mathbf{u}, w)$ is a semi-norm on $\mathbf{H}^1 \times H^2$ that satisfies the condition that if \mathbf{u}, w defines a rigid body motion, as explained in the introduction, and $f(\mathbf{u}, w) = 0$ then $\mathbf{u} = 0$ and $w = 0$. We generalize this inequality to piecewise functions on Ω_h . For Koiter shell problems. we also need to use the discrete space H_h^2 that is composed of piecewise H^2 functions with the norm defined by (1.14).

For $(\mathbf{u}, w) \in \mathbf{H}_h^1 \times H_h^2$, we define a norm

$$(4.2) \quad \|(\mathbf{u}, w)\|_{H_h^K} = \left(\sum_{\alpha=1,2} \|u_\alpha\|_{H_h^1}^2 + \|w\|_{H_h^2}^2 \right)^{1/2}.$$

Let $f(\mathbf{u}, w)$ be a semi-norm that is continuous with respect to this norm such that there is a C that only depends on the shell mid-surface and the regularity \mathcal{K} of the triangulation \mathcal{T}_h

$$(4.3) \quad |f(\mathbf{u}, w)| \leq C \|(\mathbf{u}, w)\|_{H_h^K} \quad \forall (\mathbf{u}, w) \in \mathbf{H}_h^1 \times H_h^2.$$

We assume that f satisfies the rigid body motion condition that if $(\mathbf{u}, w) \in \mathbf{H}^1 \times H^2$ defines a rigid body motion and $f(\mathbf{u}, w) = 0$ then $\mathbf{u} = 0$ and $w = 0$. We define the discrete energy norm on the space H_h^K .

$$(4.4) \quad \|(\mathbf{u}, w)\|_{E_h^K} = \left[\sum_{\alpha,\beta=1,2} \|\rho_{\alpha\beta}^K(\mathbf{u}, w)\|_{0,\Omega_h}^2 + \sum_{\alpha,\beta=1,2} \|\gamma_{\alpha\beta}(\mathbf{u}, w)\|_{0,\Omega_h}^2 \right. \\ \left. + \sum_{e \in \mathcal{E}_h^0} \left(\sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket u_\alpha \rrbracket^2 + \sum_{\alpha=1,2} h_e^{-1} \int_e \llbracket \partial_\alpha w \rrbracket^2 + h_e^{-1} \int_e \llbracket w \rrbracket^2 \right) + f^2(\mathbf{u}, w) \right]^{1/2}.$$

We then have the following generalization of the Korn's inequality for Koiter shells to piecewise functions.

Theorem 4.1. *There exists a constant C that could be dependent on the shell mid-surface, and the shape regularity \mathcal{K} of \mathcal{T}_h , but otherwise independent of the triangulation, such that*

$$(4.5) \quad \|(\mathbf{u}, w)\|_{H_h^K} \leq C \|(\mathbf{u}, w)\|_{E_h^K} \quad \forall w \in H_h^2, u_\alpha \in H_h^1.$$

Proof. Koiter's shell model is a restriction of the Naghdi's model on the subspace of zero-shear deformations. The discrete Korn's inequality (4.5) for Koiter shell can be derived from the discrete Korn's inequality for Naghdi shell (3.4).

For $u_\alpha \in H_h^1$ and $w \in H_h^2$, we define piecewise function $\theta_\alpha \in H_h^1$ by

$$(4.6) \quad \theta_\alpha = -\partial_\alpha w - b_\alpha^\gamma u_\gamma.$$

We then have $\rho_{\alpha\beta}(\boldsymbol{\theta}, \mathbf{u}, w) = -\rho_{\alpha\beta}^K(\mathbf{u}, w)$ and $\tau_\alpha(\boldsymbol{\theta}, \mathbf{u}, w) = 0$. There are constants C_1 and C_2 only depending on components of the mixed curvature tensor b_α^γ such that for all $e \in E_h^0$ and for all $\tau \in \mathcal{T}_h$

$$\begin{aligned} C_1 \left(\sum_{\alpha=1,2} \int_e \llbracket u_\alpha \rrbracket^2 + \sum_{\alpha=1,2} \int_e \llbracket \partial_\alpha w \rrbracket^2 \right) &\leq \sum_{\alpha=1,2} \int_e \llbracket u_\alpha \rrbracket^2 + \sum_{\alpha=1,2} \int_e \llbracket \theta_\alpha \rrbracket^2 \\ &\leq C_2 \left(\sum_{\alpha=1,2} \int_e \llbracket u_\alpha \rrbracket^2 + \sum_{\alpha=1,2} \int_e \llbracket \partial_\alpha w \rrbracket^2 \right), \end{aligned}$$

$$\begin{aligned} C_1 \left(\sum_{\alpha=1}^2 \|u_\alpha\|_{1,\tau} + \|w\|_{2,\tau} \right) &\leq \sum_{\alpha=1}^2 \|\theta_\alpha\|_{1,\tau} + \sum_{\alpha=1}^2 \|u_\alpha\|_{1,\tau} + \|w\|_{1,\tau} \\ &\leq C_2 \left(\sum_{\alpha=1}^2 \|u_\alpha\|_{1,\tau} + \|w\|_{2,\tau} \right). \end{aligned}$$

From these, we get the equivalences

$$(4.7) \quad C_1 \|(\mathbf{u}, w)\|_{H_h^K} \leq \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{H_h} \leq C_2 \|(\mathbf{u}, w)\|_{H_h^K}$$

and

$$(4.8) \quad C_1 \|(\mathbf{u}, w)\|_{E_h^K} \leq \|(\boldsymbol{\theta}, \mathbf{u}, w)\|_{E_h} \leq C_2 \|(\mathbf{u}, w)\|_{E_h^K}.$$

The continuity (4.3) implies the condition (3.2). The inequality (4.5) then follows from the Korn's inequality for Naghdi shell (3.4). \square

As an example for the semi-norm satisfying the continuity condition (4.3), we take

$$f(\mathbf{u}, w) = \left[\sum_{e \in \mathcal{E}_h^S \cup \mathcal{E}_h^D} \left(\int_e \sum_{\alpha=1,2} u_\alpha^2 + \int_e w^2 \right) + \sum_{e \in \mathcal{E}_h^D} \int_e (D_{\mathbf{n}} w)^2 \right]^{1/2}.$$

It follows from Theorem 2.2 that there is a C only dependent on \mathcal{K} such that the continuity condition (4.3) is satisfied by this f . Under the assumption that the measure of $\partial^D \Omega$ is positive, it is verified in [5] that if $(\mathbf{u}, w) \in \mathbf{H}^1 \times H^2$ defines a rigid body motion and $f(\mathbf{u}, w) = 0$ then $\mathbf{u} = 0$ and $w = 0$.

REFERENCES

- [1] R.A. Adams, Sobolev spaces, *Academic Press*, New York, 1975.
- [2] D.N. Arnold, F. Brezzi, Locking free finite element methods for shells, *Math. Comp.*, 66 (1997), pp. 1-14.
- [3] D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.*, 39 (2002), pp. 1749-1779.
- [4] M. Bernadou, Finite element methods for thin shell problems, *John Wiley & Sons - Masson*, 1996.
- [5] M. Bernadou, P.G. Ciarlet, B. Miara, Existence theorems for two-dimensional linear shell theories, *J. Elasticity*, 34 (1994), pp. 111-138.
- [6] S.C. Brenner, Korn's inequalities for piecewise H^1 vector fields, *Math. Comp.*, 73(2003) pp. 1067-1087.
- [7] A. Buffa, C. Ortner, Compact embeddings of broken Sobolev spaces and applications, *IMA J. Numer. Anal.*, 29(2009) pp. 827-855.
- [8] D. Chapelle, K.J. Bathe, The finite element analysis of shells – Fundamentals, *Springer*, 2011.
- [9] P.G. Ciarlet, The finite element method for elliptic problems, *North-Holland*, 1978.
- [10] D. A. Di Pietro, A. Ern, Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations, *Math. Comp.*, 79(2010) pp. 1303-1330.
- [11] K. Feng, On the theory of discontinuous finite elements, *Math. Numer. Sinica*, 4 (1979), pp. 378-385.
- [12] V. Girault, P-A. Raviart, Finite element methods for Navier–Stokes equations, theory and algorithms, *Springer-Verlag*, 1986.
- [13] W.T. Koiter, On the foundations of the linear theory of thin elastic shells, *Nederl. Akad. Wetensch. Proc. Ser. B*, 73(1970), pp. 169-195.
- [14] G. Duvaut, J.L. Lions, Inequalities in mechanics and physics, *Springer-Verlag*, 1976.
- [15] P.M. Naghdi, The theory of shells and plates, *Handbuch der Physik Vol. VIa/2*, *Springer-Verlag, Berlin*, 1972, pp. 425-640.
- [16] S. Zhang, Analysis of a discontinuous Galerkin method for Koiter shell, arXiv:1403.7052.
- [17] S. Zhang, A discontinuous Galerkin method for the Naghdi shell model, arXiv:1405.1343.